QUIZ 2 - CALCULUS 2 (2020/12/3)

1. (8 pts) Evaluate $\int \sin x \, \cos x \, e^{\sin^2 x} \, dx$.

   Solution:

$$\begin{aligned}
\int \sin x \cos x e^{\sin^2 x} dx &\overset{(u=\sin x)}{=} \int u e^{u^2} u' dx \quad (1 \text{ pt}) \\
&= \int u e^{u^2} du \quad (1 \text{ pt}) \\
&\overset{(v=u^2)}{=} \int e^v \frac{v'}{2} du \quad (1 \text{ pt}) \\
&= \frac{1}{2} \int e^v dv \quad (1 \text{ pt}) \\
&= \frac{1}{2} e^v (2 \text{ pts}) + C (1 \text{ pt}) = \frac{1}{2} e^{\sin^2 x} (1 \text{ pt}) + C.
\end{aligned}$$

2. (12 pts) Let

$$f(x) = \begin{cases} \dfrac{1}{1+x^2} & \text{if } 0 \le x \le 1 \\ \\ \dfrac{2-x}{2x} & \text{if } x > 1 \end{cases}$$

   Let $\mathcal{R}$ be the region enclosed by the x-axis, the y-axis and $y = f(x)$.

   (a) (6 pts) Find the area of $\mathcal{R}$.

   (b) (6 pts) Let S be the solid obtained by rotating $\mathcal{R}$ about the y-axis. Find the volume of S.

   Solution:

   (a) $y = f(x)$ intersects the x-axis at $x = 2$ (1 pt), so

$$\begin{aligned}
\text{Area of } \mathcal{R} &= \int_0^2 f(x) dx \\
&= \int_0^1 \frac{1}{1+x^2} dx (1 \text{ pt}) + \int_1^2 \frac{2-x}{2x} dx (1 \text{ pt}) \\
&= \int_0^1 \frac{1}{1+x^2} dx + \int_1^2 \frac{1}{x} - \frac{1}{2} dx \\
&= \arctan(x)\Big|_0^1 (1 \text{ pt}) + \left(\ln|x| \ (\text{or } \ln x) \ - \frac{1}{2} x \right)\Big|_1^2 (1 \text{ pt}) \\
&= \frac{\pi}{4} \ (\text{or } \arctan 1) + \ln 2 - \frac{1}{2}. \quad (1 \text{ pt})
\end{aligned}$$

(b) Sol 1:

$$\begin{aligned}
\text{Volume} &= \int_0^2 2\pi x f(x)dx \\
&= 2\pi\left(\int_0^1 \frac{x}{1+x^2}dx + \int_1^2 \frac{2-x}{2}dx\right) \quad (1\text{ pt}) \\
&\overset{(u=1+x^2)}{=} 2\pi\left(\int_0^1 \frac{1}{u}\frac{u'}{2}dx + \frac{1}{2}\int_1^2 (2-x)dx\right) \\
&= \pi\left(\int_1^2 \frac{1}{u}du (2\text{ pts, one for the range}) + \int_1^2 (2-x)dx\right) \\
&= \pi\left(\ln|u| \ (\text{or } \ln u)\ \Big|_1^2 (1\text{ pt}) + (2x - \frac{1}{2}x^2)\Big|_1^2 (1\text{ pt})\right) \\
&= (\ln 2 + \frac{1}{2})\pi. \quad (1\text{ pt})
\end{aligned}$$

Sol 2: $f(x)$ is strictly decreasing on $[0,2]$ (by sketching the graph or computing $f'(x)$), so the cross-section of S and planes perpendicular to the y-axis is a circle. (1 pt, need explanation) We have

$$A(y) = \begin{cases} \pi\dfrac{4}{(2y+1)^2} & \text{if } 0 \leq y \leq \dfrac{1}{2} \\[2ex] \pi\dfrac{1-y}{y} & \text{if } \dfrac{1}{2} \leq y \leq 1 \end{cases} \qquad (1\text{ pt})$$

So

$$\begin{aligned}
\text{Volume} &= \int_0^1 A(y)dy \\
&= \pi\left(\int_0^{\frac{1}{2}} \frac{4}{(2y+1)^2}dy + \int_{\frac{1}{2}}^1 \frac{1}{y} - 1dy\right) \quad (1\text{ pt}) \\
&= \pi\left(-\frac{2}{2y+1}\Big|_0^{\frac{1}{2}} (1\text{ pt}) + (\ln|y| \ (\text{or } \ln y) - y)\Big|_{\frac{1}{2}}^1 (1\text{ pt})\right) \\
&= \pi(1 + (-\ln\frac{1}{2} - \frac{1}{2})) = (\ln 2 + \frac{1}{2})\pi. \quad (1\text{ pt})
\end{aligned}$$